Frontiers of Network Science
Fall 2023

Class 14 Robustness, the rest of it
(Chapter 8 in Textbook)

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based on slides by
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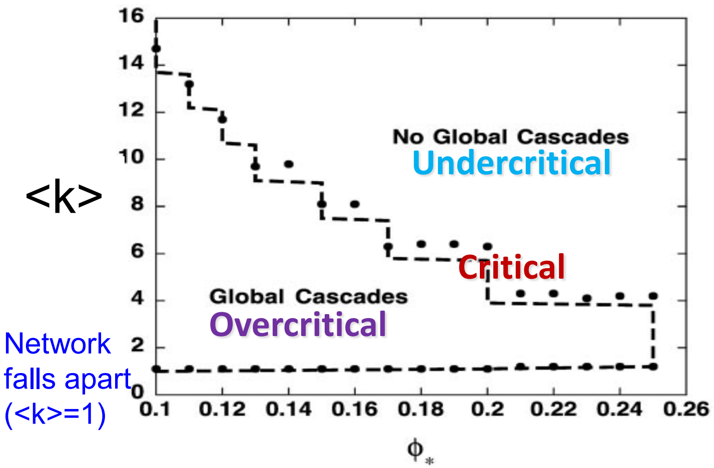
FAILURE PROPAGATION MODEL



Initial Setup

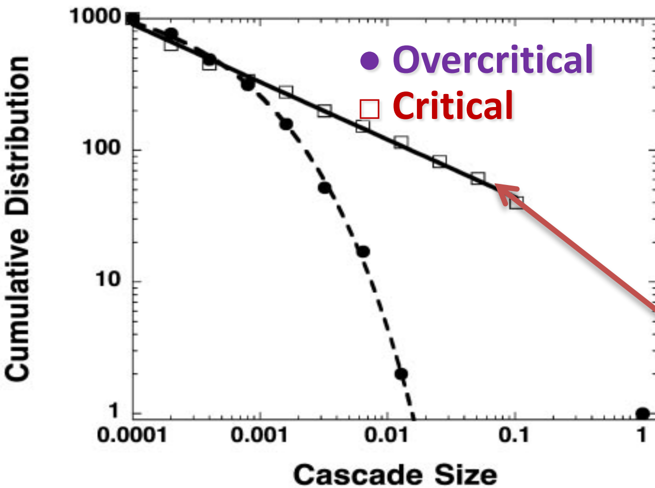
•Random graph with *N* nodes

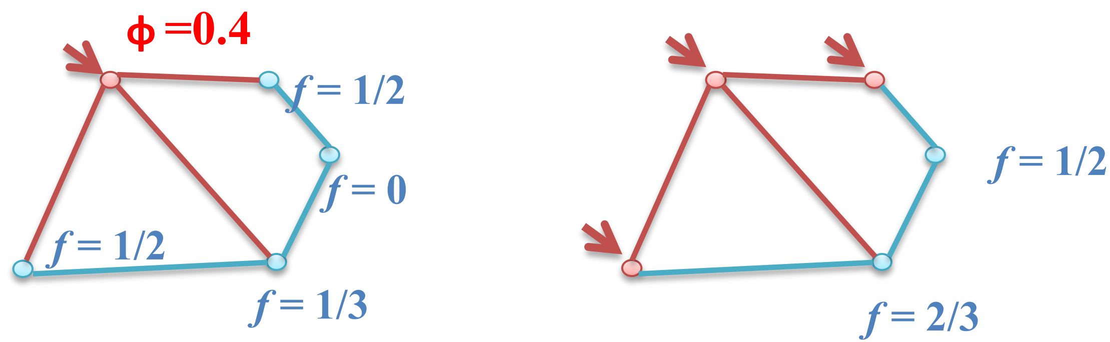
•Initially each node is functional.

Cascade

•Initiated by the failure of one node.

•f_i : fraction of failed neighbors of node *i*. Node *i* fails if f_i is greater than a global threshold ϕ.



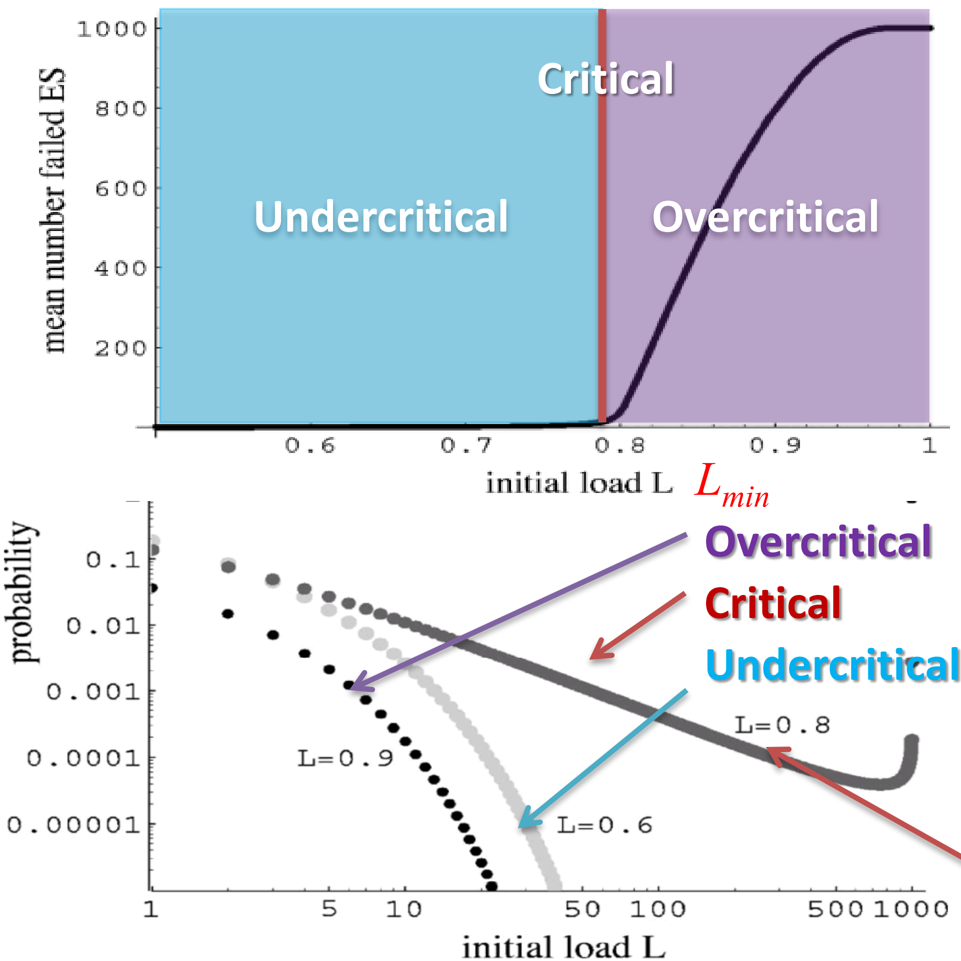


Erdos-Renyi network

$$P(S) \sim S^{-3/2}$$

D. Watts, *PNAS* 99, 5766-5771 (2002)

OVERLOAD MODEL

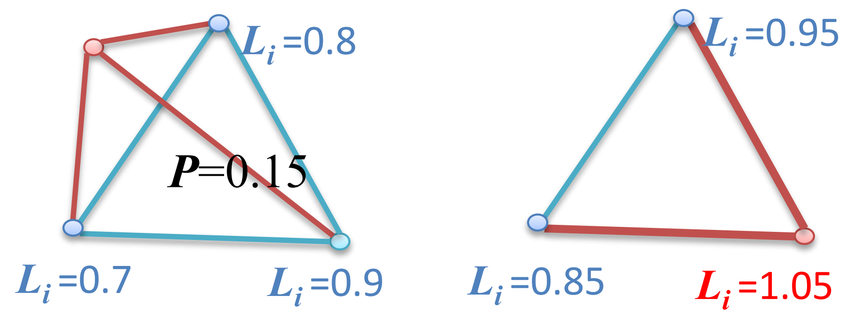


Initial Conditions

- *N* Components (complete graph)
- Each components has random initial load L_i drawn at random uniformly from $[L_{min}, 1]$.

Cascade

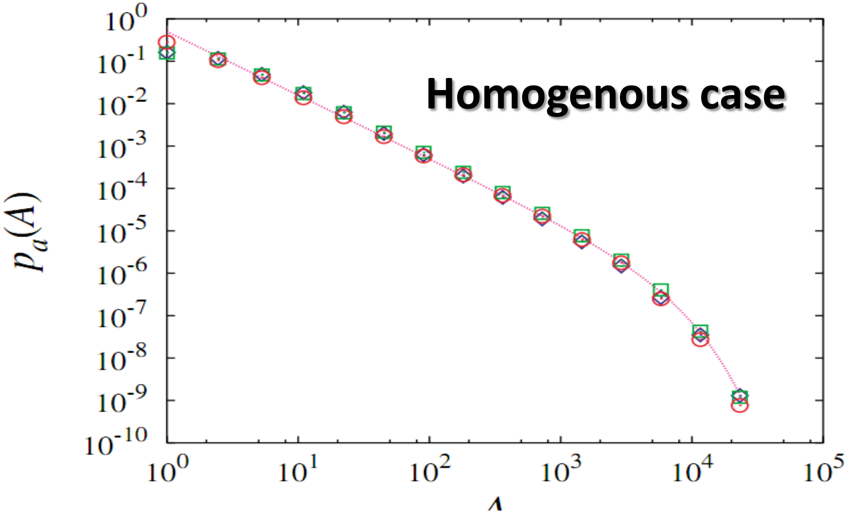
- Initiated by the failure of one component.
- Component fail when its load exceeds *1*.
- When a component fails, a fixed amount *P* is transferred to all the rests.

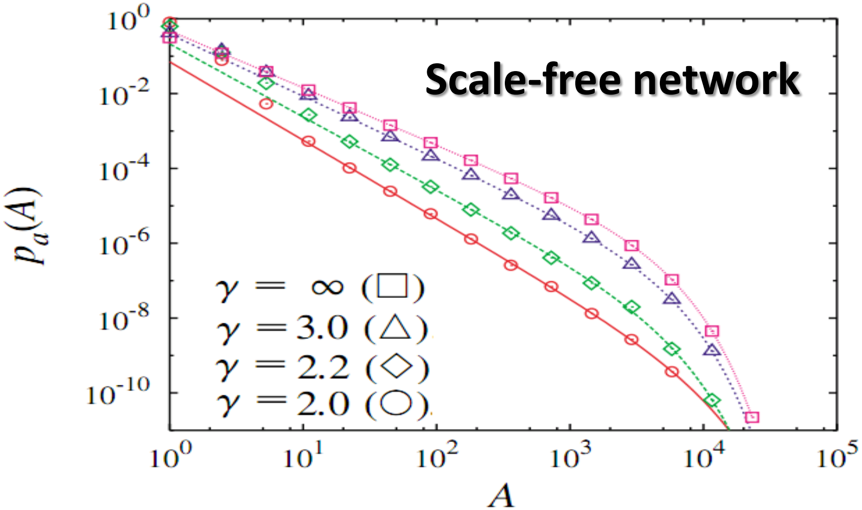


$$P(S) \sim S^{-3/2}$$

I. Dobson, B. A. Carreras, D. E. Newman, *Probab. Eng. Inform. Sci.* **19**, 15-32 (2005)

SELF-ORGANIZED CRITICALITY AKA SANDPILE MODEL





Initial Setup

- Random graph with N nodes
- Each node *i* has height $h_i = 0$.

Cascade

- At each time step, a grain is added at a randomly chosen node *i*: $h_i \leftarrow h_i + 1$
- If the height at the node *i* reaches a prescribed threshold $z_i = k_i$, then it becomes unstable and all the grains at the node topple to its adjacent nodes: $h_i = 0$ and $h_j \leftarrow h_j + 1$
- if *i* and *j* are connected.

Homogenous network: $<k^2>$ converges

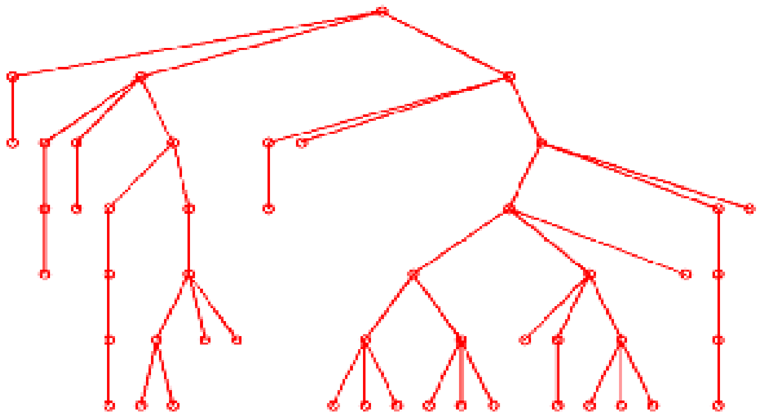
$$P(S) \sim S^{-3/2}$$

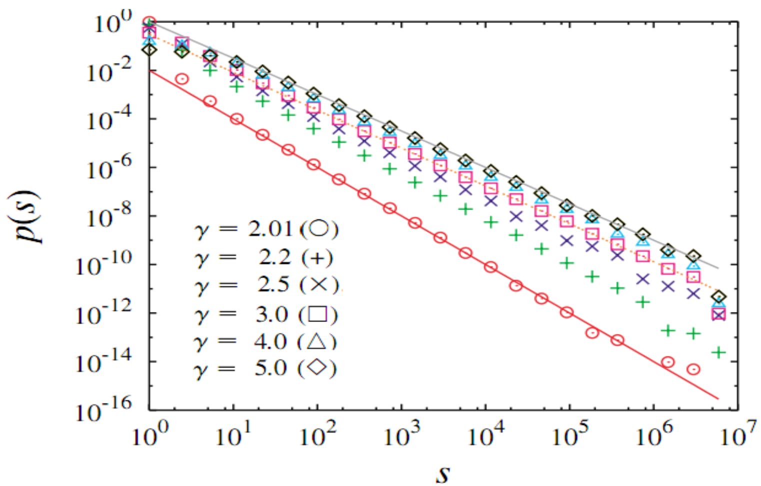
Scale-free network : $p_k \sim k^{-\gamma}$ $(2<\gamma<3)$

$$P(S) \sim S^{-\gamma/(\gamma-1)}$$

K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **91**, 148701 (2003)

BRANCHING PROCESS MODEL





Branching Process

Starting from a initial node, each node in generation *t* produces *k* number of offspring nodes in the next *t* + 1 generation, where *k* is selected randomly from a fixed probability distribution $q_k = p_{k-1}$.

Hypothesis

- No loops (tree structure)
- No correlation between branches

Fix <k>=1 to be critical → power law P(S)

Narrow distribution: $\langle k^2 \rangle$ converged

$$P(S) \sim S^{-3/2}$$

Fat tailed distribution: $q_k \sim k^{-\gamma}$ $(2<\gamma<3)$

$$P(S) \sim S^{-\gamma/(\gamma-1)}$$

K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **91**, 148701 (2003)

SHORT SUMMARY OF MODELS: UNIVERSALITY

Models	Networks	Exponents
Failure Propagation Model	ER	1.5
Overload Model	Complete Graph	1.5
BTW Sandpile Model	ER/SF	1.5 (ER) $\gamma/(\gamma - 1)$(SF)
Branching Process Model	ER/SF	1.5 (ER) $\gamma/(\gamma - 1)$(SF)

Universal for homogenous networks

$$P(S) \sim S^{-3/2}$$

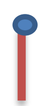
Same exponent for percolation too
(random failure, attacking, etc.)

EXPLANATION OF THE 3/2 UNIVERSALITY

Simplest Case: $q_0 = q_2 = 1/2$, $\langle k \rangle = 1$

S: number of nodes

X: number of open branches



$S = 1$

$X = 1$

½ chance

$S = S+1$

$X = X - 1$

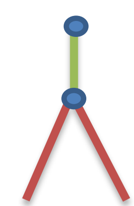
$S = 2$, $X = 0$

$k = 0$

½ chance

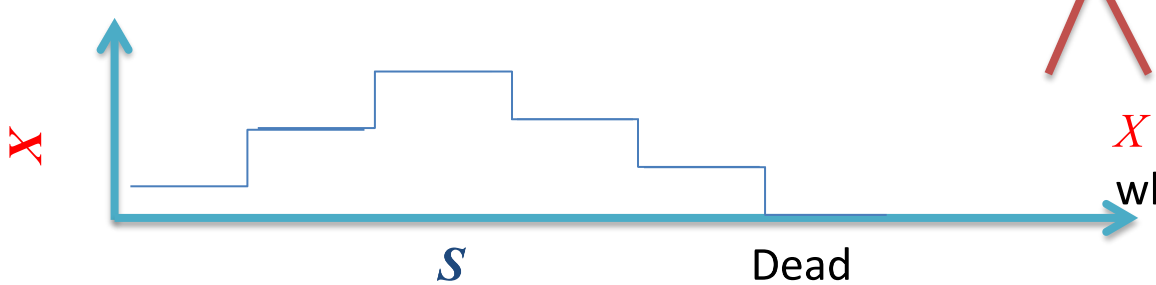
$S = S+1$

$X = X+1$



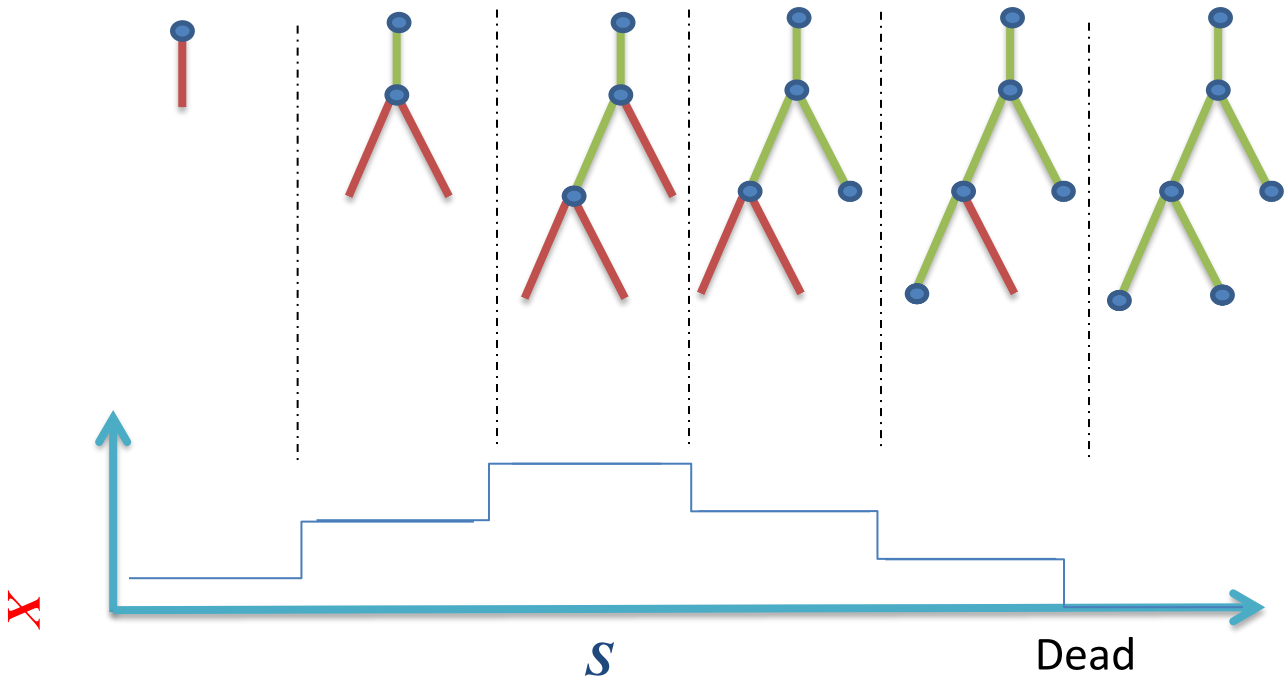
$S = 2$, $X = 2$

$k = 2$



$X > 0$, Branching process stops when $X = 0$

EXPLANATION OF THE 3/2 UNIVERSALITY



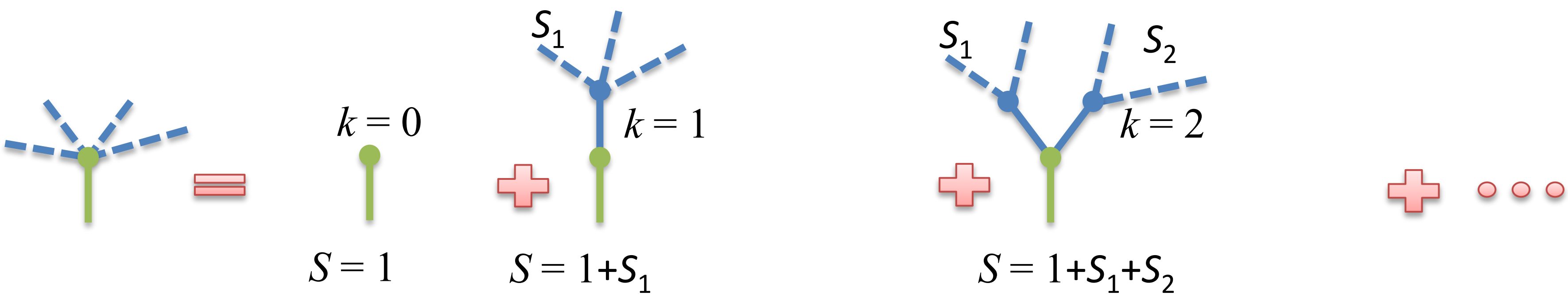
Equivalent to **1D random walk model**, where *X* and *S* are the position and time , respectively.

Question: what is the probability that *X* = 0 after *S* steps?

First return probability $\sim S^{-3/2}$

M. Ding, W. Yang, *Phys. Rev. E.* **52**, 207-213 (1995)

SIZE DISTRIBUTION OF BRANCHING PROCESS (CAVITY METHOD)



$$P(S) = q_0\delta(1) + q_1 \sum_{S_1} P(S_1)\delta(1+S_1-S) + q_2 \sum_{S_1,S_2} P(S_1)P(S_2)\delta(1+S_1+S_2-S) + \cdots$$

$$P(S) = \sum_k q_k \left(\sum_{S_1,\cdots S_k} P(S_1)P(S_2)\cdots P(S_k)\delta(1+\sum_{j=1}^{k} S_j - S) \right)$$

K.-I. Goh, D.-S. Lee, B. Kahng, and D. Kim, *Physica A* **346**, 93-103 (2005)

SOLVING THE EQUATION BY GENERATING FUNCTION

Definition:

$G_S(x) = \Sigma_{S=0}\, P(S)x^S$

$G_k(x) = \Sigma_{k=0}\, q_k x^k$

Property:

$G_S(1) = G_k(1) = 1$

$G_S'(1) = \langle S\rangle,\ G_k'(1) = \langle k\rangle$

$$P(S) = \sum_k q_k \left(\sum_{S_1,\cdots S_k} P(S_1)P(S_2)\cdots P(S_k)\delta(1+\sum_{j=1}^{k} S_j - S) \right)$$

$$G_S(x) = \sum_k q_k \left(\sum_{S_1,\cdots S_k} P(S_1)\cdots P(S_k)x^{1+\sum_j S_j} \right) = \sum_k q_k x G_S(x)^k$$
$$= xG_k(G_S(x))$$

Phase Transition

$\langle S\rangle = G_S'(1) = 1 + G_k'(1)\, G_S'(1) = 1 + \langle k\rangle \langle S\rangle$, then

$\langle S\rangle = 1/(1 - \langle k\rangle)$

The average size $\langle S\rangle$ **diverges** at $\langle k\rangle_c = 1$

FINDING THE CRITICAL EXPONENT FROM EXPANSION

Definition:

$G_S(x) = \Sigma_{S=0}\, P(S) x^S$

$G_k(x) = \Sigma_{k=0}\, q_k x^k$

Theorem:

If $P(k) \sim k^{-\gamma}$ $(2<\gamma<3)$, then for $\delta x < 0$, $|\delta x| << 1$

$G(1+\delta x) = 1 + \langle k \rangle \delta x + \langle k(k-1)/2 \rangle (\delta x)^2 + \ldots + O(|\delta x|^{\gamma - 1})$

$P(S) \sim S^{-\alpha}, 1 < \alpha < 2$

$G_S(1+\delta x) \approx 1 + A|\delta x|^{\alpha - 1}$

Homogenous case: $\langle k^2 \rangle$ converged

$\langle k \rangle = 1$, $\langle k^2 \rangle < \infty$

$G_k(1+\delta x) \approx 1 + \delta x + B\delta x^2$

Inhomogeneous case: $\langle k^2 \rangle$ diverged

$\langle k \rangle = 1$, $q_k \sim k^{-\gamma}$ $(2<\gamma<3)$

$G_k(1+\delta x) \approx 1 + \delta x + B|\delta x|^{\gamma - 1}$

CRITICAL EXPONENT FOR HOMOGENOUS CASE

Homogenous case

$$G_k(1+\delta x) \approx 1 + \delta x + B\delta x^2$$

$$G_S(1+\delta x) \approx 1 + A|\delta x|^{\alpha -1}$$

$$G_S(x) = xG_k(G_S(x))$$

$$G_S(x) \approx 1 + A|\delta x|^{\alpha -1}$$

$$\begin{aligned} xG_k(G_S(x)) &\approx (1+\delta x)[1+ (G_S(1+\delta x)-1) + B\,(G_S(1+\delta x)-1)^2] \\ &\approx (1+\delta x)[1+ A|\delta x|^{\alpha -1} + AB|\delta x|^{2\alpha -2}] \\ &= 1 + A|\delta x|^{\alpha -1} + AB|\delta x|^{2\alpha -2} + \delta x + O(|\delta x|^{\alpha}) \end{aligned}$$

The lowest order reads $AB|\delta x|^{2\alpha -2} + \delta x = 0$, which requires $2\alpha -2 = 1$ and $A = 1/B$. Or,

$$\alpha = 3/2$$

CRITICAL EXPONENT FOR INHOMOGENEOUS CASE

Inhomogeneous case

$G_k(1+\delta x) \approx 1 + \delta x + B|\delta x|^{\gamma - 1}$

$G_S(1+\delta x) \approx 1 + A|\delta x|^{\alpha - 1}$

$$G_S(x) = xG_k(G_S(x))$$

$$G_S(x) \approx 1 + A|\delta x|^{\alpha - 1}$$

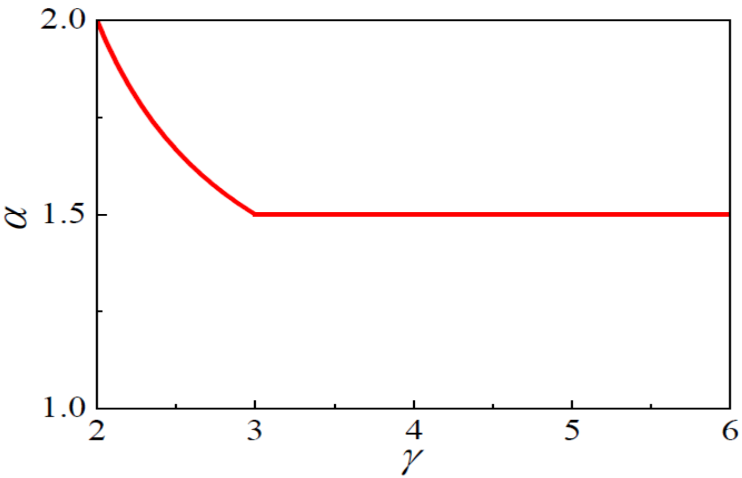
$$\begin{aligned} xG_k(G_S(x)) &\approx (1+\delta x)[1+ (G_S(1+\delta x)-1) + B\,|G_S(1+\delta x)-1|^{\gamma - 1}] \\ &\approx (1+\delta x)[1+ A|\delta x|^{\alpha - 1} + AB|\delta x|^{(\alpha - 1)(\gamma - 1)}] \\ &= 1 + A|\delta x|^{\alpha - 1} + AB|\delta x|^{(\alpha - 1)(\gamma - 1)} + \delta x + O(|\delta x|^{\alpha}) \end{aligned}$$

The lowest order reads $AB|\delta x|^{(\alpha - 1)(\gamma - 1)} + \delta x = 0$, which requires $(\alpha - 1)(\gamma - 1) = 1$ and $A = 1/B$. Or,

$$\alpha = \gamma/(\gamma - 1)$$

COMPARING THE PREDICTION WITH THE REAL DATA

$$P(S) \sim S^{-\alpha}, \alpha = \begin{cases} 3/2, & \gamma > 3 \\ \gamma/(\gamma - 1), & 2 < \gamma < 3 \end{cases}$$



Blackout

Source	Exponent	Quantity
North America	2.0	Power
Sweden	1.6	Energy
Norway	1.7	Power
New Zealand	1.6	Energy
China	1.8	Energy

Earthquake $\alpha \approx 1.67$

I. Dobson, B. A. Carreras, V. E. Lynch, D. E. Newman, *CHAOS* **17**, 026103 (2007)

Y. Y. Kagan, *Phys. Earth Planet. Inter.* **135** (2–3), 173–209 (2003)